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A NOTE ON THE DERIVATION OF THEORETICAL

AUTOCOVARIANCES FOR ARMA MODELS

by

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Derivation of the theoretical autocovariances of an ARMA model is important for a number of purposes associated with the estimation and testing of the model. One common algorithm, due to McLeod (1975), involves solving a system of linear equations. By deriving the determinant of the matrix of coefficients in these equations we can ascertain the behaviour of the algorithm with respect to the stationarity of the ARMA model.

SUMMARY

Derivation of the theoretical autocovariances of an ARMA model is important for a number of purposes associated with the estimation and testing of the model. One common algorithm, due to McLeod (1975), involves solving a system of linear equations. By deriving the determinant of the matrix of coefficients in these equations we can ascertain the behaviour of the algorithm with respect to the stationarity of the ARMA model.

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A NOTE ON THE DERIVATION OF THEORETICAL

AUTOCOVARIANCES FOR ARMA MODELS

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McLeod (1975, 1977) presents a method for deriving the theoretical autocovariance function of an ARMA model. He notes its uses in simulating ARMA processes and in deriving the asymptotic distributions of the estimated autocorrelations. The procedure can also be used in deriving ARMA model residuals (Ansley and Newbold, 1979); in obtaining the asymptotic distributions of parameter estimates and residual autocorrelations (McLeod, 1978); and in calculating the exact likelihood function of the Gaussian ARMA model (Ljung and Box, 1979; Ansley, 1979; and Dent, 1977). Ansley (1980) and Ansley and Kohn (1982) have also extended McLeod's algorithm to vector ARMA models.

An alternative and computationally superior procedure to McLeod's in the univariate case has been proposed by Wilson (1979). It also has the advantage that the stationarity of the process may be tested directly within the algorithm generating the autocovariances. The procedure for maximum likelihood estimation proposed by Dent (1977) also incorporates a test of stationarity, as do some others in that a Cholesky decomposition of the generated covariance matrix is later derived. However, this is not general.

The purpose of this note is to examine the behaviour of the McLeod algorithm with respect to stationarity.

Consider the ARMA(p,q) process $\{X_t\}$ given by

$$X_t - \phi_1 X_{t-1} - \dots - \phi_p X_{t-p} = a_t - \theta_1 a_{t-1} - \dots - \theta_q a_{t-q} . \tag{1}$$

If $p = 0$, the process is always stationary. If $p > 0$ the process is stationary if and only if the roots of the polynomial equation

$$\sum_{k=0}^{p} \phi_k z^{p-k} = 0 \tag{2}$$

all lie within the unit circle. For later reference, denote the roots of (2) by $z_1, z_2, \dots, z_p$, and denote the polynomial $\sum_{k=0}^{p} \phi_k z^k$ by $\phi(z)$.

If $p > 0$ the variance and the first $r$ autocovariances $(\gamma_0, \gamma_1, \ldots, \gamma_r)$, where $r = \max(p,q)$, are obtained by solving a system of linear equations. The matrix of coefficients of these equations, $A$ say, is given in McLeod (1975) and Ljung and Box (1979). Our interest is in how the stationarity of (1) affects the solution of these equations. We determine this by expressing $|A|$, the determinant of $A$, in terms of the roots of (2).

Consider the matrix $A(t)$ obtained by replacing $\phi_i$ by $\phi_i t^i$ in $A$. The $i^{th}$ row of $A(t)$ may be expressed as the sum of two row vectors, thus:

$$\underline{a}_i = (\phi_{i-1} t^{i-1}, \ldots, \phi_0, 0 \ldots 0)$$
$$+ (0, \phi_i t^i, \ldots, \phi_p t^p, 0, \ldots 0) \;,\; i = 1,2,\ldots,r+1 \;.$$

Clearly, $\underline{a}_i \underline{1} = \phi(t)$, for $i = 1,2,\ldots,r+1$, where $\underline{1}$ is the unit vector $(1,1,\ldots,1)'$. Hence, $\phi(t)$ is an eigenvalue of $A(t)$ and so a factor of $|A(t)|$. Similarly, $\phi(-t)$ is a factor of $|A(t)|$. We may note for later that $\phi(t)\phi(-t) = \sum_{i=1}^{p} (1 - z_i^2 t^2)$.

Suppose $z (\neq \pm 1)$ is any solution of (2), and $\underline{\zeta} = (1, z, \ldots, z^r)'$. We can use (2) to write $\underline{a}_i \underline{\zeta}$ in the form $\sum_{j=1}^{p+1-i} \phi_{j-1+i}(z^j - z^{-j})$, $i = 1,2,\ldots,r+1$. Note that if $r = p$, $\underline{a}_p \underline{\zeta} = 0$. Suppose now that $z^{-1}$ is also a solution of (2), and $\underline{\zeta}_1 = (1, z^{-1}, \ldots, z^{-r})'$. Then, $A(\underline{\zeta} + \underline{\zeta}_1) = 0$ which is possible if and only if $A$ is singular. Thus, $(1 - z_i z_j)$ is a factor of $|A|$. Further, if $\phi_i$ is replaced by $\phi_i t^i$ in (2) the roots become $\{t z_i : i = 1,2,\ldots,p\}$, and we can deduce that $(1 - z_i z_j t^2)$ is a factor of $|A(t)|$. Thus, $P(t) = \prod_{i=1}^{p} \prod_{j=i}^{p} (1 - z_i z_j t^2)$ is a factor of $|A(t)|$.

Note that $P(t)$ is a polynomial of degree $p(p+1)$ in $t$. If $r = p$, the $(k, p+2-k)$th element of $A(t)$ has the form $(\phi_p t^p$ + terms of lower power) for $k = 1,2,\ldots,p+1$. Thus, $|A(t)|$ is also a polynomial of degree $p(p+1)$. If $r = q$ then $A(t)$ has the form

$$A(t) = \begin{vmatrix} A_p(t) & 0 \\ & \\ B(t) & L(t) \end{vmatrix}$$

where $L(t)$ is a lower triangular matrix with units along the main diagonal. Thus, $|A(t)| = |A_p(t)|$ and $A_p(t)$ is $(p+1) \times (p+1)$ and has the property ascribed to $A(t)$ when $r = p$. Hence, in both cases, $|A(t)|$ is a polynomial in $t$ of degree $p(p+1)$. Further, $|A(0)| = 1 = P(0)$. Thus, taking $t = 1$, we have shown that

$$|A| = \prod_{i=1}^{p} \prod_{j=i}^{p} (1 - z_i z_j) \tag{3}$$

where $z_1, z_2, \ldots, z_p$ are the solutions of (2).

## CONCLUSIONS

From (3) it is clear that $|A| = 0$ if and only if

either (i) there is a root on the unit circle;

or (ii) there are a pair of roots symmetric about the unit circle, i.e. $z$ and $z^{-1}$.

It is comforting to know that the procedure will fail and no autocovariances will be generated when the process is non-stationary for either of the reasons given. On the other hand, it is clear that a non-stationary process which does not satisfy either (i) or (ii) will yield a set of "autocovariances". It may be possible to detect this at once, e.g. $\gamma_0$ may be negative or less than $\gamma_k$ in magnitude for some $k$. In general, however, these values can be shown to be spurious only by showing that the corresponding covariance matrix is not positive definite.

This may be achieved in a routine manner within the overall procedure. Dent (1977) suggests a check on the singular value decomposition of the covariance matrix and Pagano (1973) discusses a modified Cholesky decomposition. However, if the purpose of the procedure is estimation we may have to generate autocovariances from different sets of parameters a large number of times. In such a case an algorithm such as that proposed by Wilson, which checks stationarity while it generates autocovariances, would clearly be preferable.

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